VII.—*On the Theory of the Perturbations of the Planets.* *By* James Ivory, *A.M. F.R.S. Instit. Reg. Sc. Paris. Corresp. et Reg. Sc. Gottin. Corresp.*

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THE perturbations of the planets is the subject of reiterated researches by all the great geometers who have raised up Physical Astronomy to its present elevation. They have been successful in determining the variations which the elements of the orbit of a disturbed planet undergo; and in expressing these variations analytically, in the manner best adapted for computation. But the inquirer who turns his attention to this branch of study will find that it is made to depend upon a theory in mechanics, which is one of considerable analytical intricacy, known by the name of the Variation of the Arbitrary Constants. Considerations similar to those employed in this theory were found necessary in Physical Astronomy from its origin; but the genius of Lagrange imagined and completed the analytical processes of general application. In a dynamical problem which is capable of an exact solution, such as a planet revolving by the central attraction of the sun, the formulas constructed by Lagrange enable us to ascertain the alterations that will be induced on the original motions of the body, if we suppose it urged by new and very small forces, such as the irregular attractions of the other bodies of the planetary system. General views of this nature are very valuable, and contribute greatly to the advancement of science. But their application is sometimes attended with inconvenience. In particular cases, the general structure of the formulas may require a long train of calculation, in order to extricate the values of the quantities sought. It may be necessary for attaining this end to pass through many differential equations, and to submit to much subordinate calculation. The remedy for this inconvenience seems to lie in separating the general principles from the analytical processes by which they are carried into effect. In some important problems, a great advantage, both in brevity and clearness, will be obtained by adapting the investigation to the particular circumstance of the case, and attending solely to the principles of the method in deducing the solution. It may therefore become a question whether it be not possible to simplify physical astronomy by calling in the aid only of the usual principles of dynamics, and by setting aside every formula or equation not absolutely necessary for arriving at the final results. The utility of such an attempt, if successful, can hardly be doubted. By rendering more accessible a subject of great interest and importance, the study of English mathematicians may be recalled to a theory which, although it originated in England, has not received the attention it deserves, and which it *has* met with in foreign countries.

The paper which I have the honour to submit to the Royal Society, contains a complete determination of the variable elements of the elliptic orbit of a disturbed planet, deduced from three differential equations that follow readily from the mechanical conditions of the problem. In applying these equations, the procedure is the same whether a planet is urged by the sole action of the central force of the sun, or is besides disturbed by the attraction of other bodies revolving about that luminary; the only difference being that, in the first case, the elements of the orbit are all constant, whereas in the other case they are all variable. The success of the method here followed is derived from a new differential equation between the time and the area described by the planet in its momentary plane, which greatly shortens the investigation by making it unnecessary to consider the projection of the orbit. But the solution in this paper, although no reference is made to the analytical formulas of the theory of the variation of the arbitrary constants, is no less an application of that method, and an example of its utility and of the necessity of employing it in very complicated problems.

1. If S represent the sun and P, P′ two planets circulating round that luminary, it is proposed to investigate the effect of the attraction of P′ to disturb the motion of P and to change the elements of its orbit. We here confine our attention to one disturbing planet; for there is no difficulty in extending to any number, the conclusions that shall be established in the case of one.

The positions of the planets P and P′ may be ascertained as usual by the rectangular coordinates x, y, z and x', y', z'; x, y, x', y' being contained in a plane passing through the origin of the coordinates placed in the sun's centre; and z, z' being perpendicular to the same plane.

Further, let M, m, m' denote the respective masses of S, P, P'; r and r' the distance of P and P' from S, and ρ the distance between the two planets; then, putting $\mu = M + m$, the direct attraction between S and P will be $\frac{\mu}{r^2}$; and the resolved parts of this force, acting in the respective directions of x, y, z, and tending to diminish these lines, will be

$$\frac{\mu x}{r^3}, \quad \frac{\mu y}{r^3}, \quad \frac{\mu z}{r^3}.$$

The planet P' attracts S with a force $= \frac{m'}{r'^2}$, of which the resolved parts are,

$$\frac{m' x'}{r'^3}, \quad \frac{m' y'}{r'^3}, \quad \frac{m' z'}{r'^3}.$$

The same planet P' attracts P with the force $\frac{m'}{\rho^2}$, of which the partial forces are

$$\frac{m'(x' - x)}{\rho^3}, \quad \frac{m'(y' - y)}{\rho^3}, \quad \frac{m'(z' - z)}{\rho^3}.$$

Were S and P attracted by P' in like directions with equal intensity, the relative situation of the two bodies would not be changed, and the action of P' might be neglected: but the attractions parallel to the coordinates being unequal, the differences of these attractions, viz.

$$\frac{m'(x' - x)}{\rho^3} - \frac{m' x'}{r'^3}, \quad \frac{m'(y' - y)}{\rho^3} - \frac{m' y'}{r'^3}, \quad \frac{m'(z' - z)}{\rho^3} - \frac{m' z'}{r'^3},$$

are exerted in altering the place of P relatively to S. These last forces increase the coordinates x, y, z; and, therefore, they must be subtracted from the former forces which have opposite directions, in order to obtain the total forces acting in the directions of the coordinates and affecting the motion of P relatively to S, viz.

$$\frac{m x}{r^3} - \frac{m'(x' - x)}{\rho^3} + \frac{m' x'}{r'^3},$$

$$\frac{m y}{r^3} - \frac{m'(y' - y)}{\rho^3} + \frac{m' y'}{r'^3},$$

$$\frac{m z}{r^3} - \frac{m'(z' - z)}{\rho^3} + \frac{m' z'}{r'^3}.$$

But, if dt represent the element of the time supposed to flow uniformly, the actual velocities with which the coordinates increase are, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$; and the increments of these velocities, $\frac{ddx}{dt^2}, \frac{ddy}{dt^2}, \frac{ddz}{dt^2}$, are the effects produced by all the forces that urge the planet. Equating now the forces really in action to the measure of the effects they produce, and observing that the two equivalent quantities have been estimated in opposite directions, we obtain the following equations for determining the place of P relatively to S at any proposed instant of time,

$$\frac{ddx}{dt^2} + \frac{\mu x}{r^3} = \frac{m'(x'-x)}{\varrho^3} - \frac{m'x'}{r'^3},$$

$$\frac{ddy}{dt^2} + \frac{\mu y}{r^3} = \frac{m'(y'-y)}{\varrho^3} - \frac{m'y'}{r'^3},$$

$$\frac{ddz}{dt^2} + \frac{\mu z}{r^3} = \frac{m'(z'-z)}{\varrho^3} - \frac{m'z'}{r'^3}.$$

If we now assume

$$R = \frac{m'}{\mu} \times \left\{ \frac{1}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}} - \frac{xx' + yy' + zz'}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} \right\},$$

it will be found that the partial differentials, $\mu \times \frac{dR}{dx}, \mu \times \frac{dR}{dy}, \mu \times \frac{dR}{dz}$, are respectively equal to the quantities on the right sides of the last equations, that is, to the disturbing forces tending to increase the coordinates x, y, z. These equations may therefore be thus written,

$$\left.\begin{aligned} \frac{ddx}{\mu\, dt^2} + \frac{x}{r^3} &= \frac{dR}{dx}, \\ \frac{ddy}{\mu\, dt^2} + \frac{y}{r^3} &= \frac{dR}{dy}, \\ \frac{ddz}{\mu\, dt^2} + \frac{z}{r^3} &= \frac{dR}{dz}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad \text{(A)}$$

If it be asked, What notion must be affixed to the symbol $\mu\, dt^2$?, it will be recollected that $\frac{\mu}{r^2}$ is the attraction between S and P at the distance r; and if we suppose that P describes a circle, of which unit is the radius, round S, the centripetal force in the circle will be $\frac{\mu}{1^2}$ or μ; and the velocity with which P moves in the circle will be proportional to $\sqrt{\mu}$. Thus the algebraic quantities $t\sqrt{\mu}$ and $dt\sqrt{\mu}$ represent the arcs of this circular orbit, which are described in the times t and dt.

It is requisite in what follows to transform the coordinates x, y, z into other variable quantities better adapted for use in astronomy. Let λ and λ' denote the longitudes of the planets P and P′ reckoned in the fixt plane of $x\,y$, and s and s' the tangents of their latitudes, that is, of the angles which the radii vectores r and r' make with the same plane: then,

$$x = \frac{r\cos\lambda}{\sqrt{1+s^2}}, \qquad x' = \frac{r'\cos\lambda'}{\sqrt{1+s'^2}},$$

$$y = \frac{r\sin\lambda}{\sqrt{1+s^2}}, \qquad y' = \frac{r'\sin\lambda'}{\sqrt{1+s'^2}},$$

$$z = \frac{r\,s}{\sqrt{1+s^2}}, \qquad z' = \frac{r'\,s'}{\sqrt{1+s'^2}}.$$

In the transformations alluded to, the quantities $\frac{dR}{dx}, \frac{dR}{dy}, \frac{dR}{dz}$ must be expressed in the partial differentials of R relatively to the new variables r, λ, s; and it will conduce to clearness of method if these calculations be dispatched here. We have the equation,

$$\frac{dR}{dx} = \frac{dR}{dr}\cdot\frac{dr}{dx} + \frac{dR}{d\lambda}\cdot\frac{d\lambda}{dx} + \frac{dR}{ds}\cdot\frac{ds}{dx};$$

and having computed the differentials $\frac{dr}{dx}, \frac{d\lambda}{dx}, \frac{ds}{dx}$ from the formulas

$$r = \sqrt{x^2+y^2+z^2}, \qquad \tan.\lambda = \frac{y}{x}, \qquad s = \frac{z}{\sqrt{x^2+y^2}},$$

the substitution of the results will make known the expression of $\frac{dR}{dx}$. By the like procedure the values of $\frac{dR}{dy}$ and $\frac{dR}{dz}$ will be found

$$\left.\begin{aligned}
\frac{dR}{dx} &= \frac{dR}{dr}\cdot\frac{\cos\lambda}{\sqrt{1+s^2}} - \frac{dR}{d\lambda}\cdot\frac{\sin\lambda\sqrt{1+s^2}}{r} - \frac{dR}{ds}\cdot\frac{\cos\lambda\, s\sqrt{1+s^2}}{r},\\
\frac{dR}{dy} &= \frac{dR}{dr}\cdot\frac{\sin\lambda}{\sqrt{1+s^2}} + \frac{dR}{d\lambda}\cdot\frac{\cos\lambda\sqrt{1+s^2}}{r} - \frac{dR}{ds}\cdot\frac{\sin\lambda\, s\sqrt{1+s^2}}{r},\\
\frac{dR}{dz} &= \frac{dR}{dr}\cdot\frac{s}{\sqrt{1+s^2}} \qquad\qquad\qquad\qquad\quad + \frac{dR}{ds}\cdot\frac{\sqrt{1+s^2}}{r}.
\end{aligned}\right\}\ldots\text{(B)}$$

The new partial differentials of R represent the disturbing forces reduced to new directions. By combining the formulas (B), we get

$$\frac{d\,\mathrm{R}}{d\,r} = \frac{d\,\mathrm{R}}{d\,x} \cdot \frac{\cos\lambda}{\sqrt{1+s^2}} + \frac{d\,\mathrm{R}}{d\,y} \cdot \frac{\sin\lambda}{\sqrt{1+s^2}} + \frac{d\,\mathrm{R}}{d\,z} \cdot \frac{s}{\sqrt{1+s^2}};$$

and it will readily appear that the coefficients of $\frac{d\,\mathrm{R}}{d\,x}$, $\frac{d\,\mathrm{R}}{d\,y}$, $\frac{d\,\mathrm{R}}{d\,z}$ are the respective cosines of the angles which the directions of the forces make with r; so that $\frac{d\,\mathrm{R}}{d\,r}$ is the sum of the three partial forces that urge the planet from the sun. In like manner it may be proved that $\frac{d\,\mathrm{R}}{d\,\lambda} \cdot \frac{\sqrt{1+s^2}}{r}$ is the disturbing force perpendicular to the plane passing through the sun and the coordinate z, that is, to the circle of latitude; and that $\frac{d\,\mathrm{R}}{d\,s} \cdot \frac{1+s^2}{r}$ is the force acting in the same plane perpendicular to r, and tending to increase the latitude.

2. If the equations (A), after being multiplied by $2\,d\,x$, $2\,d\,y$, $2\,d\,z$, be added together, and then integrated, we shall get this well-known result,

$$\frac{d\,x^2 + d\,y^2 + d\,z^2}{\mu \,.\, d\,t^2} - \frac{2}{r} + \frac{1}{a} = 2\int d'\,\mathrm{R}, \quad \ldots \ldots \ldots \quad (1)$$

in which $\frac{1}{a}$ is the arbitrary constant, and the symbol $d'\,\mathrm{R}$ is put for

$$\frac{d\,\mathrm{R}}{d\,x}\,d\,x + \frac{d\,\mathrm{R}}{d\,y}\,d\,y + \frac{d\,\mathrm{R}}{d\,z}\,d\,z;$$

that is, for the differential of R, on the supposition that x, y, z, the coordinates of the disturbed planet, are alone variable. If we conceive that R is transformed into a function of the other quantities r, λ, s, we shall therefore have

$$d'\,\mathrm{R} = \frac{d\,\mathrm{R}}{d\,r}\,d\,r + \frac{d\,\mathrm{R}}{d\,\lambda}\,d\,\lambda + \frac{d\,\mathrm{R}}{d\,s}\,d\,s.$$

Supposing that the radius vector r, at the end of the small interval of time $d\,t$, becomes equal to $r + d\,r$, and that $d\,v$ expresses the small angle contained between r and $r + d\,r$, we shall have

$$d\,r^2 + r^2\,d\,v^2 = d\,x^2 + d\,y^2 + d\,z^2;$$

for each of these quantities is equal to the square of the small portion of its orbit which the planet describes in the time dt. The last equation may therefore be thus written,

$$\frac{dr^2}{\mu\, dt^2} + \frac{r^2 dv^2}{\mu\, dt^2} - \frac{2}{r} + \frac{1}{a} = 2\int d' \mathrm{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The double of the small area contained between the radii r and $r + dr$, is equal to $r^2 dv$; and as x, y, z and $x + dx$, $y + dy$, $z + dz$, are the coordinates of the extremities of the radii, the projections of the area upon the planes of xy, xz, yz, are respectively equal to

$$x\,dy - y\,dx, \quad x\,dz - z\,dx, \quad y\,dz - z\,dy:$$

wherefore, according to a well known property, we shall have,

$$\frac{r^4 dv^2}{\mu\, dt^2} = \frac{(x\,dy - y\,dx)^2}{\mu\, dt^2} + \frac{(x\,dz - z\,dx)^2}{\mu\, dt^2} + \frac{(y\,dz - z\,dy)^2}{\mu\, dt^2}:$$

and the differential of this equation, dt being constant, may be thus written,

$$d\,.\,\frac{r^4 dv^2}{\mu\, dt^2} = 2\,(x^2 + y^2 + z^2)\,.\left(\frac{dx\,ddx + dy\,ddy + dz\,ddz}{\mu\, dt^2}\right)$$

$$- 2\,(x\,dx + y\,dy + z\,dz)\,.\left(\frac{x\,ddx + y\,ddy + z\,ddz}{\mu\, dt^2}\right).$$

Now, substitute the values of the second differentials taken from the equations (A), and we shall obtain, first,

$$\frac{dx\,ddx + dy\,ddy + dz\,ddz}{\mu\, dt^2} = \frac{d\mathrm{R}}{dx}\,dx + \frac{d\mathrm{R}}{dy}\,dy + \frac{d\mathrm{R}}{dz}\,dz - \frac{dr}{r^2} = d'\mathrm{R} - \frac{dr}{r^2}:$$

and, secondly,

$$\frac{x\,ddx + y\,ddy + z\,ddz}{\mu\, dt^2} = \frac{d\mathrm{R}}{dx}\,x + \frac{d\mathrm{R}}{dy}\,y + \frac{d\mathrm{R}}{dz}\,z - \frac{1}{r} = \frac{d\mathrm{R}}{dr}\,r - \frac{1}{r}:$$

wherefore, since $x^2 + y^2 + z^2 = r^2$ and $x\,dx + y\,dy + z\,dz = r\,dr$, the foregoing differential equation will become by substitution,

$$d\,.\,\frac{r^4 dv^2}{\mu\, dt^2} = 2\,r^2\left(d'\mathrm{R} - \frac{d\mathrm{R}}{dr}\,dr\right),$$

or, which is equivalent,

$$d\,.\,\frac{r^4 dv^2}{\mu\, dt^2} = 2\,r^2\left(\frac{d\mathrm{R}}{d\lambda}\,d\lambda + \frac{d\mathrm{R}}{ds}\,ds\right).$$

By integrating,

$$\left.\begin{aligned} &r^2\,dv = h\,dt\sqrt{\mu}, \\ &h^2 = h_0^2 + 2\int r^2\left(d'\,\mathrm{R} - \frac{d\,\mathrm{R}}{d\,r}\,dr\right), \\ &h^2 = h_0^2 + 2\int r^2\left(\frac{d\,\mathrm{R}}{d\,\lambda}\,d\,\lambda + \frac{d\,\mathrm{R}}{d\,s}\,d\,s\right), \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

the constant h_0 being equal to $\frac{r^2\,d\,v}{d\,t\,\sqrt{\mu}}$ when $t = 0$.

Further, let the first of the equations (A) multiplied by y be subtracted from the second multiplied by x; then

$$\frac{d\,.\,(x\,dy - y\,dx)}{\mu\,d\,t^2} = \frac{d\,\mathrm{R}}{dy}\,x - \frac{d\,\mathrm{R}}{d\,x}\,y:$$

and, by converting the quantities in this equation into functions of r, λ, s,

$$\frac{d\,.\left(\frac{r^2}{1+s^2}\cdot\frac{d\,\lambda}{d\,t\,\sqrt{\mu}}\right)}{d\,t\,\sqrt{\mu}} = \frac{d\,\mathrm{R}}{d\,\lambda}:$$

and by multiplying both sides by $2\,.\,\frac{r^2}{1+s^2}\,.\,d\,\lambda$,

$$d\,.\left(\frac{r^2}{1+s^2}\cdot\frac{d\,\lambda}{d\,t\,\sqrt{\mu}}\right)^2 = 2\,\frac{r^2}{1+s^2}\cdot\frac{d\,\mathrm{R}}{d\,\lambda}\,d\,\lambda:$$

and, by integrating,

$$\left.\begin{aligned} &\frac{r^2\,d\,\lambda}{1+s^2} = h'\,d\,t\sqrt{\mu}, \\ &h'^2 = h_0'^2 + 2\int\frac{r^2}{1+s^2}\cdot\frac{d\,\mathrm{R}}{d\,\lambda}\,d\,\lambda, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (4)$$

h_0' being a constant.

The equations that have been investigated, which are only three, the first and second being one equation in two different forms, are sufficient for determining the place of a planet at any proposed instant of time, whether it revolves solely by the central force of the sun, or is disturbed by the irregular attractions of the other bodies of the system. The second and third equations ascertain the form and magnitude of the orbit in its proper plane, and the place of the planet; the fourth equation enables us to find the angle in which the plane of the orbit is inclined to the immoveable plane of $x\,y$, and the position of the line in which the two planes intersect one another.

3. We begin with the more simple case of the problem, when the planet is urged solely by the central force of the sun. On this supposition, there being no disturbing forces, we must make $R = 0$ in the equations of the last §. By the formulas (3) and (4), we have,

$$r^2\,d\,v = h\,d\,t\sqrt{\mu},$$

$$\frac{r^2}{1+s^2}\,.\,d\,\lambda = h'\,d\,t\sqrt{\mu}\,;$$

and h, h', are constant quantities. Now $\frac{r}{\sqrt{1+s^2}}$ is the projection of r upon the plane of $x\,y$; and the area $\frac{r^2}{1+s^2}\,.\,d\,\lambda$ is the projection of the area $r^2\,d\,v$ upon the same plane; wherefore, if i denote the angle of inclination which the plane containing the radii vectores r and $r + d\,r$, has to the plane of $x\,y$, we shall have

$$\cos i = \frac{\frac{r^2}{1+s^2}\,.\,d\,\lambda}{r^2\,d\,v} = \frac{h'}{h}\,;$$

which proves that a plane passing through the sun's centre and any two places of the planet infinitely near one another, has constantly the same inclination to the immoveable plane of $x\,y$. And it further proves that the planet moves in one invariable plane; for, unless this were the case, the areas described round the sun in any consecutive small portions of time, could not constantly have the same proportion to their projections upon the plane of $x\,y$.

The orbit in its proper plane will be determined by the equations (2) and (3), viz.

$$\frac{d\,r^2}{\mu\,d\,t^2} + \frac{r^2\,d\,v^2}{\mu\,d\,t^2} - \frac{2}{r} + \frac{1}{a} = 0,$$

$$r^2\,d\,v = h\,d\,t\sqrt{\mu},$$

a and h being arbitrary quantities. By exterminating $d\,t\sqrt{\mu}$ from the first equation,

$$h^2\,.\,\frac{d\,r^2}{r^4\,d\,v^2} + \frac{h^2}{r} - \frac{2}{r} + \frac{1}{a} = 0;$$

by multiplying all the terms by $\frac{r^2}{a}$, and adding 1 to both sides,

$$\frac{h^2}{a} \cdot \frac{d\,r^2}{r^2\,d\,v^2} + \left(1 - \frac{r}{a}\right)^2 = 1 - \frac{h^2}{a};$$

and by introducing the new quantity e^2,

$$e^2 = 1 - \frac{h^2}{a}$$

$$(1 - e^2)\frac{d\,r^2}{r^2\,d\,v^2} + \left(1 - \frac{r}{a}\right)^2 = e^2.$$

This equation is solved by assuming

$$\frac{d\,r}{r\,d\,v} = \frac{e \sin \theta}{1 + e \cos \theta},$$

$$1 - \frac{r}{a} = e \times \frac{\cos \theta + e}{1 + e \cos \theta},$$

the arc θ remaining indeterminate. For, if the assumed quantities be substituted, the equation will be verified, and the arc θ will be eliminated. In order to determine θ, let the second of the formulas be differentiated, and equate $\frac{dr}{r}$ to the like value in the first formula; then,

$$dv = d\,\theta; \text{ and } v - \varpi = \theta.$$

The nature of the orbit is therefore determined by these two equations,

$$\frac{d\,r}{r\,d\,v} = \frac{e \sin (v - \varpi)}{1 + e \cos (v - \varpi)},$$

$$r = \frac{a\,(1 - e^2)}{1 + e \cos (v - \varpi)}:$$

the first of which shows that the two conditions $\frac{d\,r}{d\,v} = 0$, and $\sin (v - \varpi) = 0$, must take place at the same time; so that ϖ is the place of the planet when its distance from the sun is a minimum $= a\,(1 - e)$, or a maximum $= a\,(1 + e)$: and the second proves that the orbit of the planet is an ellipse having the sun in one focus; a being the mean distance; e the eccentricity; and $v - \varpi$ the true anomaly, that is, the angular distance from the perihelion or aphelion;

from the perihelion if e be positive, and from the aphelion if the same quantity be negative.

It must however be observed that the preceding determination rests entirely on the assumption that, in the equation $e^2 = 1 - \frac{h^2}{a}$, the quantity $\frac{h^2}{a}$ is positive and less than unit. Without entering upon any detail, which our present purpose does not require, all the possible cases of the problem will be succinctly distinguished by writing the equation in this form,

$$\frac{h^2}{1+e} = (1-e) \times a.$$

The quantity on the left side being essentially positive, the two factors on the other side must both have the same sign. If they are positive, the orbit will be an ellipse; if they are negative, and consequently e greater than unit, the curve described by the body will be a hyperbola; and it will be a parabola, when $e = 1$, and a and $1-e$ pass from being positive to be negative, at which limit the equation will assume this form,

$$\frac{h^2}{2} = 0 \times \infty.$$

In all the cases $\frac{h^2}{1+e}$ is the perihelion distance.

The nature of the orbit being found, we have next to determine the relation between the time and the angular motion of the planet. For this purpose we have the equation, $r^2\,dv = h\,dt\sqrt{\mu}$, from which, by substituting the values of r and h, we deduce

$$\frac{dt\sqrt{\mu}}{a^{\frac{3}{2}}} = \frac{(1-e^2)^{\frac{3}{2}}\,dv}{\left(1 + e\cos(v-\varpi)\right)^2}.$$

Let $\frac{\sqrt{\mu}}{a^{\frac{3}{2}}} = n$; then, by integrating,

$$nt + \varepsilon - \varpi = \int \frac{(1-e^2)^{\frac{3}{2}}\,dv}{\left(1 + e\cos(v-\varpi)\right)^2},$$

the quantity under the sign of integration being taken so as to vanish when $v - \varpi = 0$, and ε being a constant quantity. The mean motion of the planet reckoned from a given epoch, is equal to $nt + \varepsilon$; and the mean anomaly, to

$n t + \varepsilon - \varpi$, the true anomaly being $v - \varpi$. The equation may be put in this form,

$$n t + \varepsilon - \varpi = \int \frac{\sqrt{1 - e^2} \, . \, d v}{1 + e \cos (v - \varpi)} - e \times \frac{\sqrt{1 - e^2} \, . \sin (v - \varpi)}{1 + e \cos (v - \varpi)}:$$

and, if we assume

$$\sin u = \frac{\sqrt{1 - e^2} \, . \sin (v - \varpi)}{1 + e \cos (v - \varpi)}, \quad \cos u = \frac{\cos (v - \varpi) + e}{1 + e \cos (v - \varpi)};$$

we shall find,

$$u = \int \frac{\sqrt{1 - e^2} \, . \, d v}{1 + e \cos (v - \varpi)}:$$

so that we readily arrive at these results,

$$n t + \varepsilon - \varpi = u - e \sin u,$$

$$r = \frac{a (1 - e^2)}{1 + e \cos (v - \varpi)} = a (1 - e \cos u),$$

$$\tan \frac{v - \varpi}{2} = \tan \frac{u}{2} \times \sqrt{\frac{1 + e}{1 - e}}.$$

These last are the formulas that occur in the solution of KEPLER's problem, the arc u being the anomaly of the eccentric. Having found the expression of the eccentric anomaly in terms of the mean anomaly from the first of the formulas, we thence deduce the true anomaly $v - \varpi$, and the radius vector r, for any proposed instant of time. The analytical solution of these questions is omitted; the sole intention of treating here of the motion of a planet circulating by the central force of the sun, being to elucidate the investigations that are to follow respecting the orbit of a disturbed planet.

The purposes of astronomy require further that the motion of the planet in its orbit be connected with the longitudes and latitudes estimated with regard to the immovable plane of $x y$. The orbit being supposed to intersect the immovable plane, and the angle of inclination being represented by i, let N stand for the longitude of the ascending node, and P for the place of the same node in the plane of the orbit and reckoned from the same origin with the true motion v: then $v - \mathrm{P}$, or the distance of the planet from the node in the plane of the orbit, is the hypothenuse of a right-angled spherical triangle, one side of which is the arc $\lambda - N$ in the immovable plane, and the remaining side is the latitude having s for its tangent: wherefore we have

$$\tan(\lambda - N) = \tan(v - P)\cos i,$$

$$s = \tan i \sin(\lambda - N).$$

The first of these equations enables us to compute λ when v is given, and conversely; by means of the second, the latitude is found. The practical calculations are much facilitated by expressing the quantities sought in converging serieses: but the discussion of these points is beside our present purpose.

4. We now proceed to investigate the effect of the disturbing force of the planet P′ in altering the orbit of P. For this purpose we have the equations (3) and (4), viz.

$$r^2\,dv = h\,dt\sqrt{\mu},$$

$$\frac{r^2}{1+s^2}\,.\,d\lambda = h'\,dt\sqrt{\mu};$$

of which the first is the expression of the small area described round the sun by the planet in the time dt, and the other is the projection of that area upon the immovable plane of xy. Wherefore, if i denote the angle of inclination which the plane passing through the sun and the radii vectores r and $r + dr$, has to the plane of xy, we shall have

$$\cos i = \frac{\frac{r^2}{1+s^2}\,.\,d\lambda}{r^2\,dv} = \frac{h'}{h}:$$

and, as h' and h vary incessantly by the action of the disturbing forces, it follows that the momentary plane in which the planet moves is continually changing its inclination to the fixed plane. Let i' be the value of i when $t = 0$; then $\cos i' = \frac{h'_0}{h_0}$; and, by the formulas (3) and (4), we shall have,

$$h^2 = h_0^2 + 2\int r^2\left(\frac{dR}{d\lambda}\,d\lambda + \frac{dR}{ds}\,ds\right),$$

$$h'^2 = h_0^2\cos^2 i' + 2\int r^2\,.\,\frac{dR}{d\lambda}\,.\,\frac{d\lambda}{1+s^2},$$

$$h''^2 = h^2 - h'^2 = h_0^2\sin^2 i' + 2\int r^2\left(\frac{dR}{d\lambda}\,.\,\frac{s^2\,d\lambda}{1+s^2} + \frac{dR}{ds}\,ds\right):$$

and hence, in consequence of what has been shown,

$$\cos^2 i = \frac{h'^2}{h^2}; \ \sin^2 i = \frac{h''^2}{h^2}; \ \tan^2 i = \frac{h''^2}{h'^2}.$$

Let the momentary plane of the planet's orbit, that is, the plane passing through r and $r + dr$, intersect the immovable plane of xy, and put N for the place of the ascending node: then s and $s + ds$ will be the tangents of the latitudes at the distances $\lambda - \mathrm{N}$, and $\lambda + d\lambda - \mathrm{N}$ from the node: and, i being the angle contained between the two planes, we shall have,

$$\left.\begin{aligned} s &= \tan i \sin(\lambda - \mathrm{N}), \\ \frac{ds}{d\lambda} &= \tan i \cos(\lambda - \mathrm{N}). \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (5)$$

By adding the squares of these equations,

$$s^2 + \frac{ds^2}{d\lambda^2} = \tan^2 i = \frac{h''^2}{h'^2};$$

by differentiating, making $d\lambda$ constant,

$$h'^2\left(\frac{dds}{d\lambda^2} + s\right) = \frac{h''\,dh'' - h'\,dh'\left(s^2 + \frac{ds^2}{d\lambda^2}\right)}{ds};$$

and, by substituting the values of $h''\,dh''$ and $h'\,dh'$,

$$\frac{dds}{d\lambda^2} + s = \frac{r^2}{h'^2} \cdot \left\{\frac{d\mathrm{R}}{ds} - \frac{1}{1+s^2} \cdot \frac{d\mathrm{R}}{d\lambda} \cdot \frac{ds}{d\lambda}\right\}.$$

Since i is variable in the equations (5), it is obvious that N, or the place of the node, must likewise vary. By combining each of the two equations with the differential of the other, these results will be obtained,

$$0 = \frac{d \cdot \tan i}{d\lambda} \sin(\lambda - \mathrm{N}) - \frac{d\mathrm{N}}{d\lambda} \tan i \cos(\lambda - \mathrm{N})$$

$$\frac{dds}{d\lambda^2} + s = \frac{d \cdot \tan i}{d\lambda} \cos(\lambda - \mathrm{N}) + \frac{d\mathrm{N}}{d\lambda} \tan i \sin(\lambda - \mathrm{N});$$

from which we deduce,

$$di = \cos^2 i \cos(\lambda - \mathrm{N}) \cdot \left\{\frac{dds}{d\lambda^2} + s\right\} \cdot d\lambda,$$

$$d\mathrm{N} = \frac{\cos i \sin(\lambda - \mathrm{N})}{\sin i} \cdot \left\{\frac{dds}{d\lambda^2} + s\right\} \cdot d\lambda;$$

and, by substituting the value of $\frac{d\,d\,s}{d\,\lambda^2} + s$,

and, observing that $r^2\,d\,\lambda = (1 + s^2)\,h'\,d\,t\sqrt{\mu} = (1 + s^2)\,h\cos i\,d\,t\sqrt{\mu}$, we finally get,

$$\left.\begin{aligned} d\,i &= \cos i \cos(\lambda - \mathrm{N}) \,.\, \left\{(1 + s^2)\frac{d\,\mathrm{R}}{d\,s} - \frac{d\,\mathrm{R}}{d\,\lambda} \cdot \frac{d\,s}{d\,\lambda}\right\} \cdot \frac{d\,t\sqrt{\mu}}{h} \\ d\,\mathrm{N} &= \frac{\sin(\lambda - \mathrm{N})}{\sin i} \cdot \left\{(1 + s^2)\frac{d\,\mathrm{R}}{d\,s} - \frac{d\,\mathrm{R}}{d\,\lambda} \cdot \frac{d\,s}{d\,\lambda}\right\} \cdot \frac{d\,t\sqrt{\mu}}{h} \end{aligned}\right\} \quad \ldots \quad (6)$$

These equations determine the motion of the node in longitude, and the variation in the inclination of the orbit. They are rigorously exact, and may be transformed in various ways, as it may suit the purpose of the inquirer.

We proceed now to investigate the motion of the planet round the sun. For this purpose we have the equations (2) and (3), viz.

$$\frac{d\,r^2}{\mu\,d\,t^2} + \frac{r^2\,d\,v^2}{\mu\,d\,t^2} - \frac{2}{r} + \frac{1}{a} = 2\int d'\,\mathrm{R},$$

$$r^2\,d\,v = h\,d\,t\sqrt{\mu}.$$

And first, as the small arc $d\,v$ contained between the two radii r and $r + d\,r$, continually passes from one plane to another, it is requisite to inquire what notion we must affix to the sum v. The momentary plane of the planet's motion, in shifting its place, turns upon a radius vector; and if we suppose a circle concentric with the sun to be described in it, and to remain firmly attached to it, the differentials $d\,v$ will evidently accumulate upon the circumference of the circle, and will form a continuous sum, in the same manner as if the plane remained motionless in one position. The arc v is therefore the angular motion of the planet round the sun in the moveable plane, and is reckoned upon the circumference of the circle from an arbitrary origin.

In the first of the foregoing equations a is an arbitrary constant, and I shall put,

$$\frac{1}{\mathrm{a}} = \frac{1}{a} - 2\int d'\,\mathrm{R};$$

so that we shall have

$$\frac{d\,r^2}{\mu\,d\,t^2} + \frac{r^2\,d\,v^2}{\mu\,d\,t^2} - \frac{2}{r} + \frac{1}{\mathrm{a}} = 0,$$

$$r^2\,d\,v = h\,d\,t\sqrt{\mu};$$

which are different from the corresponding equations in the last section in no respect, except that here h and a are both variable, whereas in the other case they were both constant. Treating these equations exactly as before, we first get by exterminating $dt\sqrt{\mu}$,

$$h^2 \frac{dr^2}{r^4 dv^2} + \frac{h^2}{r^2} - \frac{2}{r} + \frac{1}{\mathrm{a}} = 0;$$

then, by multiplying all the terms by $\frac{r^2}{\mathrm{a}}$ and adding 1 to both sides,

$$\frac{h^2}{\mathrm{a}} \cdot \frac{dr^2}{r^2 dv^2} + \left(1 - \frac{r}{\mathrm{a}}\right)^2 = 1 - \frac{h^2}{\mathrm{a}};$$

from which we deduce

$$e^2 = 1 - \frac{h^2}{\mathrm{a}},$$

$$(1 - e^2) \frac{dr^2}{r^2 dv^2} + \left(1 - \frac{r}{\mathrm{a}}\right)^2 = e^2.$$

The last equation is solved by the same assumptions as before, viz.

$$\frac{dr}{r\,dv} = \frac{e \sin\theta}{1 + e\cos\theta},$$

$$1 - \frac{r}{\mathrm{a}} = e \times \frac{\cos\theta + e}{1 + e\cos\theta};$$

but it must be recollected that in these formulas, a and e are both variable. By differentiating the expression of r, viz.

$$r = \frac{\mathrm{a}(1 - e^2)}{1 + e\cos\theta} = \frac{h^2}{1 + e\cos\theta},$$

we get,

$$\frac{dr}{r} = \frac{e\sin\theta \, . \, d\theta}{1 + e\cos\theta} + \frac{2\,dh}{h} - \frac{\cos\theta \, . \, de}{1 + e\cos\theta};$$

and by equating this expression to the value of $\frac{dr}{r}$ taken from the first formula, and reducing, we obtain,

$$e(dv - d\theta)\sin\theta + \cos\theta \, . \, de = \frac{2\,dh}{h}(1 + e\cos\theta).$$

It appears therefore that $v - \theta$, or ϖ, is a variable quantity; and the formulas that determine the elliptic orbit, and the variation of ϖ, are as follows:

$$\frac{dr}{r\,dv} = \frac{e \sin(v - \varpi)}{1 + e\cos(v - \varpi)}$$

$$r = \frac{\mathrm{a}(1 - e^2)}{1 + e\cos(v - \varpi)} = \frac{h^2}{1 + e\cos(v - \varpi)},$$

$$e\,d\varpi \sin(v - \varpi) + de\,.\cos(v - \varpi) = \frac{2\,dh}{h}\Big(1 + e\cos(v - \varpi)\Big). \quad . \quad . \quad . \quad (7)$$

It is obvious that this last formula is tantamount to the equating to zero of the differential of r relatively to the variables, h, e, ϖ, or a, e, ϖ; it may therefore be thus written,

$$\frac{dr}{d\mathrm{a}}d\mathrm{a} + \frac{dr}{de}de + \frac{dr}{d\varpi}d\varpi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The equations that have been investigated, enable us to deduce from the disturbing forces the variable elements of the ellipse that coincides momentarily with the real path of the planet; a being the mean distance, e the eccentricity; ϖ the place of the perihelion, and h^2 the semi-parameter. We have next to find the relation between the time and the angular motion in the variable orbit. This will be accomplished by means of the equation $r^2\,dv = h\,dt\sqrt{\mu}$; from which we obtain, by substituting the values of r and h,

$$\frac{dt\sqrt{\mu}}{\mathrm{a}^{\frac{3}{2}}} = \frac{(1 - e^2)^{\frac{3}{2}}\,.\,dv}{\Big(1 + e\cos(v - \varpi)\Big)^2}.$$

The integral $\int\frac{dt\sqrt{\mu}}{\mathrm{a}^{\frac{3}{2}}}$, supposed to commence with the time, is the mean motion of the planet: when there is no disturbing force, a being constant, the mean motion is proportional to the time and equal to $n \times t$; but the action of the disturbing forces, by making a variable, alters the case, and requires the introducing of a new symbol ζ to represent the mean motion. Thus we have

$$\zeta = \int\frac{\mu}{\mathrm{a}^{\frac{3}{2}}} \times dt; \quad d\zeta = \frac{(1 - e^2)^{\frac{3}{2}}\,.\,dv}{\Big(1 + e\cos(v - \varpi)\Big)^2}.$$

The value of ζ cannot be obtained directly by integration on account of the variability of e and ϖ. Let $f(v - \varpi, e)$ express that function of the true anomaly which is equal to the mean anomaly in the undisturbed orbit; that is, suppose,

$$f(v-\varpi, e) = \int \frac{(1-e^2)^{\frac{3}{2}} . dv}{\left(1 + e\cos(v-\varpi)\right)^2},$$

the integral being taken on the conditions that it vanishes when $v - \varpi = 0$, and that e and ϖ are constant. If now we make e and ϖ variable, we shall have,

$$\frac{d.f(v-\varpi, e)}{dv} dv + \frac{d.f(v-\varpi, e)}{de} de + \frac{d.f(v-\varpi, e)}{d\varpi} d\varpi = d.f(v-\varpi, e).$$

But the partial differential relatively to v, is no other than the expression of $d\zeta$: wherefore,

$$d\zeta + \frac{d.f(v-\varpi, e)}{de} de + \frac{d.f(v-\varpi, e)}{d\varpi} d\varpi = d.f(v-\varpi, e).$$

By introducing a new symbol this equation may be separated into the two which follow,

$$d\zeta + d\varepsilon - d\varpi = d.f(v-\varpi, e),$$

$$d\varepsilon - d\varpi = \frac{d.f(v-\varpi, e)}{de} de + \frac{d.f(v-\varpi, e)}{d\varpi} d\varpi.$$

In the integral

$$\zeta + \varepsilon - \varpi = f(v-\varpi, e),$$

$\zeta + \varepsilon$ is the mean motion of the planet reckoned from a given epoch, ε however representing a quantity that varies incessantly by the action of the disturbing forces, the amount of the variation being determined by the second formula in which the value of ε alone has not been previously ascertained. The mean anomaly of the planet is $\zeta + \varepsilon - \varpi$; and the integral shows that there is the same finite equation between the mean and the true anomalies in the disturbed orbit as when there is no disturbing force. It follows therefore that, in both the cases, the true anomaly, the true motion of the planet, and the radius vector, are deducible from the mean anomaly by the same rules and by the solution of KEPLER's problem.

In order to find the value of the new variable ε, it is necessary to eliminate the differential coefficients from its expression. Differentiating relatively to e and ϖ, we shall get,

$$\frac{d.f(v-\varpi, e)}{de} = -\sqrt{1-e^2} . \int \frac{2\cos(v-\varpi) + 3e + e^2\cos(v-\varpi)}{\left(1 + e\cos(v-\varpi)\right)^3} . dv,$$

$$\frac{d\,.f(v-\varpi,e)}{d\varpi}=-(1-e^2)^{\frac{3}{2}}\,.\int\frac{2e\sin(v-\varpi)\,.\,dv}{\left(1+e\cos(v-\varpi)\right)^3}:$$

and, by integrating,

$$\frac{d\,.f(v-\varpi,e)}{de}=-\frac{\sin(v-\varpi)\sqrt{1-e^2}}{1+e\cos(v-\varpi)}-\frac{\sin(v-\varpi)\sqrt{1-e^2}}{\left(1+e\cos(v-\varpi)\right)^2}=$$

$$-\frac{\left(2+e\cos(v-\varpi)\right)\sin(v-\varpi)\sqrt{1-e^2}}{\left(1+e\cos(v-\varpi)\right)^2},$$

$$\frac{d\,.f(v-\varpi,e)}{d\varpi}=-\frac{(1-e^2)^{\frac{3}{2}}}{\left(1+e\cos(v-\varpi)\right)^2}.$$

These values being substituted in the foregoing formula, we shall find this result, after dividing all the terms by the coefficient of $d\varpi$,

$$\frac{\left(1+e\cos(v-\varpi)\right)^2}{(1-e^2)^{\frac{3}{2}}}\,.\,(d\varepsilon-d\varpi)=-\frac{\left(2+e\cos(v-\varpi)\right)\sin(v-\varpi)}{1-e^2}\,de-d\varpi,$$

or, more concisely,

$$\frac{a^2\sqrt{1-e^2}}{r^2}\,.\,(d\varepsilon-d\varpi)=-\frac{\left(2+e\cos(v-\varpi)\right)\sin(v-\varpi)}{1-e^2}\,.\,de-d\varpi.\,.\quad .\quad .\ (9)$$

From the equation between the mean and the true anomalies we deduce,

$$v=\zeta+\varepsilon-\Phi,$$

Φ representing a function of the mean anomaly $\zeta+\varepsilon-\varpi$: and as the differentials of ζ and v are independent of the differentials of ε, e, and ϖ, we shall have,

$$\frac{dv}{d\varepsilon}d\varepsilon+\frac{dv}{de}de+\frac{dv}{d\varpi}d\varpi=0.\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\ (10)$$

Now,

$$\frac{dv}{d\varepsilon}=1-\frac{d\,.\,\Phi}{d\varepsilon};\qquad\frac{dv}{d\varpi}=-\frac{d\,.\,\Phi}{d\varpi}:$$

and, because, Φ is a function of $\varepsilon-\varpi$,

$$\frac{d\,.\,\Phi}{d\varepsilon}=-\frac{d\,.\,\Phi}{d\varpi}:\qquad\text{consequently, }\frac{dv}{d\varepsilon}+\frac{dv}{d\varpi}=1.$$

The equation may therefore be thus written,

$$\frac{dv}{d\varepsilon}(d\varepsilon - d\varpi) + \frac{dv}{de}\,de + d\varpi = 0.$$

But, v being a function $\zeta + \varepsilon$, it follows that,

$$\frac{dv}{d\varepsilon} = \frac{dv}{d\zeta} = \frac{a^2\sqrt{1-e^2}}{r^2};$$

and thus it appears that the equation we are considering is identical with the formula (9): from which we learn that,

$$\frac{dv}{de} = \frac{\left(2 + e\cos(v-\varpi)\right)\sin(v-\varpi)}{1-e^2}.$$

It remains now to say a word about the longitudes and latitudes of the planet reckoned on the immoveable plane of xy. The variable quantities N and i denote the longitude of the ascending node, and the inclination of the orbit, in respect to the fixed plane: let P represent the place of the same node on the moveable plane of the planet, this arc being reckoned from the same origin as the true motion v: then, because the momentary plane in which the planet moves, in taking a new position, turns about a radius vector, it is obvious that, if $d\mathrm{N}$ be the motion of the node in the fixed plane of xy, $\cos i \times d\mathrm{N}$ will be its motion in the variable plane of the orbit. Wherefore we have,

$$d\mathrm{P} = \cos i \times d\mathrm{N}, \text{ and } \mathrm{P} = \int \cos i \,.\, d\mathrm{N},$$

a constant being supposed to accompany the integral. This being observed, it is obvious that the same equations as in the case of the undisturbed orbit, will obtain between the quantities under consideration, viz.

$$\tan(\lambda - \mathrm{N}) = \cos i \tan(v - \mathrm{P}),$$

$$s = \tan i \sin(\lambda - \mathrm{N}).$$

The foregoing investigations prove that the motion of a disturbed planet may be accurately represented by a variable ellipse coinciding momentarily with the real path of the planet. The variations, in the magnitude, the form, and the position of the ellipse, have been expressed by equations that depend upon the disturbing forces. A new inquiry presents itself: to exhibit the differentials of the elements of the variable orbit in the forms best adapted for use.

5. The expressions of the coordinates x, y, z, in terms of the variables r, λ, s, are as follows:

$$x = \frac{r\cos\lambda}{\sqrt{1+s^2}}, \quad y = \frac{r\sin\lambda}{\sqrt{1+s^2}}, \quad z = \frac{rs}{\sqrt{1+s^2}}:$$

and, if we write $\lambda - \text{N} + \text{N}$ for λ, we shall get,

$$x = r \,.\left\{ \frac{\cos(\lambda - \text{N})}{\sqrt{1+s^2}} \cos \text{N} - \frac{\sin(\lambda - \text{N})}{\sqrt{1+s^2}} \sin \text{N} \right\},$$

$$y = r \,.\left\{ \frac{\sin(\lambda - \text{N})}{\sqrt{1+s^2}} \cos \text{N} + \frac{\cos(\lambda - \text{N})}{\sqrt{1+s^2}} \sin \text{N} \right\}.$$

But $v - \text{P}$ in the plane of the planet's motion is the hypothenuse of a right-angled spherical triangle of which $\lambda - \text{N}$ is one side, s the tangent of the other side, and i the angle opposite to this latter side; and from these considerations we get

$$\frac{\cos(\lambda - \text{N})}{\sqrt{1+s^2}} = \cos(v - \text{P}), \quad \frac{\sin(\lambda - \text{N})}{\sqrt{1+s^2}} = \sin(v - \text{P})\cos i, \text{ and}$$

$$\frac{s}{\sqrt{1+s^2}} = \sin(v - \text{P})\sin i:$$

wherefore we have these values of the coordinates,

$$x = r \,.\, \{\cos(v - \text{P})\cos \text{N} - \sin(v - \text{P})\sin \text{N}\cos i\}$$

$$y = r \,.\, \{\sin(v - \text{P})\cos \text{N}\cos i + \cos(v - \text{P})\sin \text{N}\}$$

$$z = r \,.\, \sin(v - \text{P})\sin i.$$

The radius vector r is a function of v, a, e, ϖ, viz.

$$r = \frac{\text{a}(1 - e^2)}{1 + e\cos(v - \varpi)}:$$

and thus the coordinates x, y, z, are functions of v and the five elements a, e, ϖ, N, i; for P is no independent quantity, since it varies with N. In order to abridge we may write X, Y, Z for the multipliers of r in the foregoing expressions of x, y, z; so that

$$x = r \times \text{X}, \qquad y = r \times \text{Y}, \qquad z = r \times \text{Z}.$$

Now, on account of the equation (8) we have

$$\frac{dx}{da}da + \frac{dx}{de}de + \frac{dx}{d\varpi}d\varpi = \left\{\frac{dr}{da}da + \frac{dr}{de}de + \frac{dr}{d\varpi}d\varpi\right\} \times X = 0;$$

and, in like manner,

$$\frac{dy}{da}da + \frac{dy}{de}de + \frac{dz}{d\varpi}d\varpi = 0,$$

$$\frac{dz}{da}da + \frac{dz}{de}de + \frac{dz}{d\varpi}d\varpi = 0.$$

Further, we have,

$$\frac{dx}{dN}dN + \frac{dx}{di}di = r \,.\left\{\left(\frac{dX}{dP}\cos i + \frac{dX}{dN}\right)dN + \frac{dX}{di}di\right\};$$

and, if the expression on the right side of this formula be computed, it will be found equal to

$$\{\sin(v - P)\,di - \cos(v - P)\sin i\,dN\} \times \sin N \sin i;$$

and, by substituting the values of $\sin(v - P)$ and $\cos(v - P)$, the same quantity may be thus written,

$$\{\sin(\lambda - N)\,d.\tan i - \cos(\lambda - N)\tan i\,dN\} \times \frac{\sin N \sin i \cos i}{\sqrt{1 + s^2}};$$

which expression is equal to zero in consequence of what was shown in § 4. Wherefore we have,

$$\left.\begin{array}{l} \dfrac{dx}{dN}dN + \dfrac{dx}{di}di = 0; \\ \text{and similarly,} \\ \dfrac{dy}{dN}dN + \dfrac{dy}{di}di = 0 \\ \dfrac{dz}{dN}dN + \dfrac{dz}{di}di = 0. \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (11)$$

It follows from what has been said that the expressions of dx, dy, dz contain dv only, and are independent of the differentials of the five elements, a, e, ϖ, N, i, which destroy one another and disappear. And further, if in x, y, z we substitute for v, its value in terms of the mean motion and the mean anomaly, viz.

$$v = \zeta + \varepsilon - \Phi,$$

the expressions of dx, dy, dz will contain $d\zeta$ only: for dv contains $d\zeta$ only, and is independent of the differentials of ε, e, ϖ. Thus we have

$$dx = \frac{dx}{d\zeta} d\zeta = \frac{dx}{dt} dt, \quad dy = \frac{dy}{d\zeta} d\zeta = \frac{dy}{dt} dt, \quad dz = \frac{dz}{d\zeta} d\zeta = \frac{dz}{dt} dt.$$

It is in these properties that we recognise the principle of the *Variation of the arbitrary constants.* The finite expressions of x, y, z, being the same in the immoveable ellipse described by the sole action of the centripetal force of the sun, and in the variable ellipse which represents the motion of a disturbed planet, they will verify the equations (A), supposing the arbitrary quantities constant, and neglecting the disturbing forces. The velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are the same whether the arbitrary quantities remain constant or vary; and thus, for a moment of time dt, the motion in the invariable ellipse coincides with that of the planet in its real path. But, in the next moment of time, the planet will quit the periphery of the ellipse supposed to continue invariable; because the forces in that orbit are different from the forces which urge the planet. In the immoveable ellipse the forces in the directions of the coordinates are equal to $\frac{ddx}{dt^2}$, $\frac{ddy}{dt^2}$, $\frac{ddz}{dt^2}$, the arbitrary quantities being constant; but, in the case of the planet, the like forces are equal to the same differentials augmented by the variation of the arbitrary quantities, the additions thus introduced being equal to the disturbing forces, $\mu \frac{dR}{dx}$, $\mu \frac{dR}{dy}$, $\mu \frac{dR}{dz}$. It is in this manner that an elliptic orbit, by the variation of its elements, is capable of representing at every moment of time both the velocity of a disturbed planet, and the forces by which it is urged.

And generally, when a dynamical problem admits of an exact solution, the arbitrary quantities may be made to vary so as not to alter the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$; and the additions which the variation of the same quantities makes to the expressions $\frac{ddx}{dt^2}$, $\frac{ddy}{dt^2}$, $\frac{ddz}{dt^2}$ will represent new forces introduced in the problem. By means of this artifice we may estimate the effect of any disturbing forces, more especially of such as bear an inconsiderable proportion to the principal forces, in altering the original motion of the body. This is the principle of the *Variation of the arbitrary constants*, a method which has been much discussed, and which is now probably exhausted. It originated in the first researches on physical astronomy, and has been matured in passing through the hands of EULER, LAGRANGE, LAPLACE, and POISSON. The labours of these great geometers have raised up a general analytical theory applicable to every case, and requiring no more than the substitution of the particular forces under consideration. Invaluable as are such extensive views, the application of formulas constructed on considerations of so general a nature, may not always be very ready or very direct, and may require much subordinate calculation. In important problems it may be advantageous to separate the principles of the method from the analytical processes with which they are conjoined, and to deduce the solution directly from the principles themselves by attending closely to the peculiar nature of the case.

Distinguishing the two planets by their masses m and m', the symbol R stands for a function of x, y, z, the coordinates of the disturbed planet m, and of x', y', z', those of the disturbing planet m'. The expressions of these latter coordinates will be obtained by marking all the quantities in the values of x, y, z, with an accent, understanding that the accented quantities denote the same things relatively to the orbit of m', that the unaccented quantities represent in the orbit of m. The function R may be transformed in two ways, according as we substitute, for the coordinates, one set of values or another. It will be changed into a function of the four independent quantities r, v, N, i, and of the like four accented quantities of the planet m', by substituting the values of the coordinates obtained in the beginning of this section: and in this case, for greater precision, the partial differentials of R relatively to r and v will be written with parentheses, thus, $\left(\frac{d\mathrm{R}}{dr}\right)$ and $\left(\frac{d\mathrm{R}}{dv}\right)$. When the values of x, y, z, in terms of the mean motion ζ and of the six elements, $a, \varepsilon, e, \varpi, \mathrm{N}, i$, and the like values of the other coordinates are substituted, R will be a function of the mean motions ζ and ζ', and of the respective elements of the two orbits. In this latter transformation, the partial differentials of R will be written, as usual, without parentheses. It may not be improper to set down here the expressions of such of these partial differentials as we shall have occasion to refer to,

$$\frac{d\mathrm{R}}{d\mathrm{a}} = \left(\frac{d\mathrm{R}}{dr}\right) \cdot \frac{dr}{d\mathrm{a}} = \left(\frac{d\mathrm{R}}{dr}\right) \cdot \frac{r}{\mathrm{a}},$$

$$\frac{d\mathrm{R}}{d\varepsilon} = \left(\frac{d\mathrm{R}}{dr}\right) \cdot \frac{dr}{dv} \cdot \frac{dv}{d\varepsilon} + \left(\frac{d\mathrm{R}}{dv}\right) \cdot \frac{dv}{d\varepsilon}, \qquad \text{(C)}$$

$$\frac{d\mathrm{R}}{de} = \left(\frac{d\mathrm{R}}{dr}\right) \cdot \left(\frac{dr}{de} + \frac{dr}{dv} \cdot \frac{dv}{de}\right) + \left(\frac{d\mathrm{R}}{dv}\right) \cdot \frac{dv}{de},$$

$$\frac{d\mathrm{R}}{d\varpi} = \left(\frac{d\mathrm{R}}{dr}\right) \cdot \left(\frac{dr}{d\varpi} + \frac{dr}{dv} \cdot \frac{dv}{d\varpi}\right) + \left(\frac{d\mathrm{R}}{dv}\right) \cdot \frac{dv}{d\varpi},$$

in which expressions, it need hardly be observed, that $\frac{dv}{d\varepsilon}$, $\frac{dv}{de}$, $\frac{dv}{d\varpi}$, refer to this value of v,

$$v = \zeta + \varepsilon - \Phi.$$

Proceeding now to reduce the differentials of the elements of the variable orbit to the forms best adapted for use, we have this formula for the mean distance a,

$$\frac{1}{\mathrm{a}} = \frac{1}{a} - 2\int d'\mathrm{R}: \text{ consequently, } \frac{d\mathrm{a}}{\mathrm{a}^2} = 2\,d'\mathrm{R}.$$

Now, when x, y, z are transformed into expressions of ζ and the elements of the orbit, it has been proved that dx, dy, dz contain $d\zeta$ only, and are independent of the differentials of the elements: wherefore, the value $d'\mathrm{R}$ will be found by differentiating R, making ζ the only variable, that is, we shall have,

$$d'\mathrm{R} = \frac{d\mathrm{R}}{dx}dx + \frac{d\mathrm{R}}{dy}dy + \frac{d\mathrm{R}}{dz}dz = \frac{d\mathrm{R}}{d\zeta}d\zeta.$$

But substituting this value,

$$\left.\begin{aligned} d\mathrm{a} &= 2\,\mathrm{a}^2 \frac{d\mathrm{R}}{d\zeta}\,d\zeta, \\ \mathrm{a} &= a + 2\int \mathrm{a}^2 \frac{d\mathrm{R}}{d\zeta}\,d\zeta, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (12)$$

The mean motion ζ is defined by this equation, $d\zeta = \frac{dt\sqrt{\mu}}{\mathrm{a}^{\frac{3}{2}}}$. But, we have,

$$\frac{1}{a} = \frac{1}{\mathrm{a}}(1 + 2\,\mathrm{a}\int d'\mathrm{R}); \text{ and, } \frac{dt\sqrt{\mu}}{a^{\frac{3}{2}}} = \frac{dt\sqrt{\mu}}{\mathrm{a}^{\frac{3}{2}}} \cdot (1 + 2\,\mathrm{a}\int d'\mathrm{R})^{\frac{3}{2}}.$$

Let $n^2 = \frac{\mu}{a^3}$, n being the constant of the mean motion in the primitive ellipse, when $t = 0$: then

$$\left.\begin{aligned} n\,dt &= d\zeta\left(1 + 2\,\mathrm{a}\int\frac{d\,\mathrm{R}}{d\,\zeta}\,d\,\zeta\right)^{\frac{3}{2}} \\ d\,\zeta &= n\,dt - d\,\zeta\left\{3\mathrm{a}\int\frac{d\,\mathrm{R}}{d\,\zeta}\,d\,\zeta + \tfrac{3}{2}\,\mathrm{a}^2\left(\int\frac{d\,\mathrm{R}}{d\,\zeta}\,d\,\zeta\right)^2 \&\text{c.}\right. \end{aligned}\right\} \quad \ldots \quad (13)$$

Taking next the semi-parameter h^2, we have, by equation (3),

$$h\,dh = r^2\left(d'\mathrm{R} - \frac{d\,\mathrm{R}}{d\,r}\,dr\right):$$

but $d'\mathrm{R} = \left(\frac{d\,\mathrm{R}}{d\,v}\right)\,d\,v + \left(\frac{d\,\mathrm{R}}{d\,r}\right)\,d\,r$; wherefore,

$$h\,dh = \left(\frac{d\,\mathrm{R}}{d\,v}\right).\,r^2\,dv = \mathrm{a}^2\sqrt{1-e^2}\,.\left(\frac{d\,\mathrm{R}}{d\,v}\right)\,d\,\zeta.$$

In order to find the value of $\left(\frac{d\,\mathrm{R}}{d\,v}\right)$, let the expressions of $\frac{d\,\mathrm{R}}{d\,\varepsilon}$ and $\frac{d\,\mathrm{R}}{d\,\varpi}$ in the formulas (C), be added: then, since it has been shown that $\frac{d\,v}{d\,\varepsilon} + \frac{d\,v}{d\,\varpi} = 1$, we get,

$$\frac{d\,\mathrm{R}}{d\,\varepsilon} + \frac{d\,\mathrm{R}}{d\,\varpi} = \left(\frac{d\,\mathrm{R}}{d\,r}\right)\left(\frac{d\,r}{d\,\varpi} + \frac{d\,r}{d\,v}\right) + \left(\frac{d\,\mathrm{R}}{d\,v}\right):$$

and, because r is a function of $v - \varpi$, $\frac{d\,r}{d\,v} + \frac{d\,r}{d\,\varpi} = 0$; wherefore,

$$\frac{d\,\mathrm{R}}{d\,\varepsilon} + \frac{d\,\mathrm{R}}{d\,\varpi} = \left(\frac{d\,\mathrm{R}}{d\,v}\right).$$

Further, because ε always accompanies ζ, or which is the same thing, because R is a function of $\zeta + \varepsilon$, we have $\frac{d\,\mathrm{R}}{d\,\zeta} = \frac{d\,\mathrm{R}}{d\,\varepsilon}$: consequently,

$$\frac{d\,\mathrm{R}}{d\,\zeta} + \frac{d\,\mathrm{R}}{d\,\varpi} = \left(\frac{d\,\mathrm{R}}{d\,v}\right).$$

By substituting this value,

$$\left.\begin{aligned} h\,dh &= \mathrm{a}^2\sqrt{1-e^2}\,.\left(\frac{d\,\mathrm{R}}{d\,\zeta} + \frac{d\,\mathrm{R}}{d\,\varpi}\right)d\,\zeta, \\ h^2 &= a\,(1 - e'^2) + 2\int \mathrm{a}^2\sqrt{1-e^2}\left(\frac{d\,\mathrm{R}}{d\,\zeta} + \frac{d\,\mathrm{R}}{d\,\varpi}\right)d\,\zeta, \end{aligned}\right\} \quad \ldots \quad (14)$$

the semi-parameter of the primitive ellipse being equal to $a(1-e'^2)$, and its eccentricity to e'.

The eccentricity is determined by this formula,

$$e^2 = 1 - \frac{h^2}{\mathrm{a}}:$$

by differentiating,

$$e\,d\,e = -\frac{h\,d\,h}{\mathrm{a}} + \frac{h^2}{2}\cdot\frac{d\,\mathrm{a}}{\mathrm{a}^2} = -\frac{h\,d\,h}{\mathrm{a}} + \mathrm{a}\,(1-e^2)\frac{d\,\mathrm{R}}{d\,\zeta}\,d\,\zeta:$$

and by substituting the value of $h\,d\,h$,

$$d\,e = -\,\mathrm{a}\sqrt{1-e^2}\,.\left\{\frac{1-\sqrt{1-e^2}}{e}\cdot\frac{d\,\mathrm{R}}{d\,\zeta} + \frac{d\,\mathrm{R}}{e\,d\,\varpi}\right\}d\,\zeta. \quad . \quad . \quad . \qquad (15)$$

For the variation of the perihelion we have the formula (7), which may be written in this manner,

$$\frac{2}{r}\,h\,d\,h = \cos(v-\varpi)\,d\,e + e\sin(v-\varpi)\,d\,\varpi:$$

and by multiplying all the terms by e,

$$e\sin(v-\varpi)\,.\,e\,d\,\varpi = \frac{2\,e}{r}\,h\,d\,h - \cos(v-\varpi)\,e\,d\,e:$$

and because $e\,d\,e = -\frac{h\,d\,h}{\mathrm{a}} + h^2\,d'\mathrm{R}$,

$$e\sin(v-\varpi)\,.\,e\,d\,\varpi = \left(\frac{2\,e}{r} + \frac{\cos v-\varpi}{\mathrm{a}}\right)h\,d\,h - h^2\cos(v-\varpi)\,d'\,\mathrm{R}.$$

Further, $d'\,\mathrm{R} = \frac{h\,d\,h}{r^2} + \left(\frac{d\,\mathrm{R}}{d\,r}\right)d\,r = \frac{h\,d\,h}{r^2} + \frac{1}{r^2}\cdot\frac{d\,r}{d\,v}\cdot\left(\frac{d\,\mathrm{R}}{d\,r}\right).r^2\,d\,v$:

and, by substituting this value,

$$e\sin(v-\varpi)\,.\,e\,d\,\varpi = \left(\frac{2\,e}{r} + \frac{\cos(v-\varpi)}{\mathrm{a}} - \frac{h^2\cos(v-\varpi)}{r^2}\right).\,h\,d\,h$$

$$-\frac{h^2}{r^2}\cdot\frac{d\,r}{d\,v}\cos(v-\varpi)\left(\frac{d\,\mathrm{R}}{d\,r}\right)r^2\,d\,v.$$

Now $\frac{h^2}{r^2}\cdot\frac{d\,r}{d\,v} = e\sin(v-\varpi)$; and it will be found that the coefficient of $h\,d\,h$ is equal to,

$$\frac{\left(2 + e\cos(v-\varpi)\right).\,e\sin^2(v-\varpi)}{\mathrm{a}\,(1-e^2)} = \frac{1}{\mathrm{a}}\cdot\frac{d\,v}{d\,e}\cdot e\sin(v-\varpi):$$

wherefore, by substituting and dividing all the terms by $e \sin(v - \varpi)$,

$$e\,d\varpi = \frac{1}{\mathrm{a}} \cdot \frac{dv}{de} \cdot h\,dh - \cos(v - \varpi)\left(\frac{d\mathrm{R}}{dr}\right) . r^2\,dv.$$

But $h\,dh = \left(\frac{d\mathrm{R}}{dv}\right) r^2\,dv$, and,

$$-\cos(v - \varpi) = \frac{1}{\mathrm{a}}\left(\frac{dr}{de} + \frac{dr}{dv} \cdot \frac{dv}{de}\right):$$

wherefore,

$$e\,d\varpi = \left\{\frac{dv}{de}\left(\frac{d\mathrm{R}}{dv}\right) + \left(\frac{dr}{de} + \frac{dr}{dv} \cdot \frac{dv}{de}\right)\left(\frac{d\mathrm{R}}{dr}\right)\right\} . \frac{r^2\,dv}{\mathrm{a}}:$$

and, because $r^2\,dv = \mathrm{a}^2\sqrt{1 - e^2} . d\zeta$,

$$d\varpi = \frac{\mathrm{a}\sqrt{1 - e^2}}{e} \cdot \frac{d\mathrm{R}}{de} . d\zeta. \quad \text{(16)}$$

The variation of ε, the longitude of the epoch, must be deduced from the equation (9), viz.

$$\frac{\mathrm{a}^2\sqrt{1 - e^2}}{r^2}(d\varepsilon - d\varpi) = -\frac{dv}{de}de - d\varpi.$$

From this de may be eliminated by means of the equation (7), viz.

$$\frac{2}{r}\,h\,dh = \cos(v - \varpi)\,de + e\sin(v - \varpi)\,d\varpi;$$

and the result will be

$$\frac{\mathrm{a}^2\sqrt{1 - e^2} . \cos(v - \varpi)}{r^2}(d\varepsilon - d\varpi) =$$

$$-\frac{2}{r} \cdot \frac{dv}{de}\,h\,dh - \left\{\cos(v - \varpi) - e\sin(v - \varpi)\frac{dv}{de}\right\} d\varpi.$$

Now the coefficient of $d\varpi$ is equal to

$$(1 - e^2)\frac{\mathrm{a}^2}{r^2}\cos(v - \varpi) - \frac{2\mathrm{a}e}{r};$$

wherefore, by multiplying by r^2, we get

$$\mathrm{a}^2\sqrt{1 - e^2} . \cos(v - \varpi)\,(d\varepsilon - d\varpi) = -2r . \frac{dv}{de} . h\,dh$$

$$-\,\mathrm{a}\,\{\mathrm{a}(1 - e^2)\cos(v - \varpi) - 2re\}\,d\varpi:$$

substitute now the value of $d\varpi$ in equation (16), and that of $h\,dh$ viz. $a^2\sqrt{1-e^2}\left(\frac{dR}{dv}\right)d\zeta$, then

$$\cos(v-\varpi)(d\varepsilon-d\varpi) = -2r\frac{dv}{de}\cdot\left(\frac{dR}{dv}\right)d\zeta$$
$$-\left\{\frac{a(1-e^2)\cos v-\varpi}{e}-2r\right\}\frac{dR}{de}d\zeta.$$

But, as appears from the formulas (C),

$$\frac{dv}{de}\cdot\left(\frac{dR}{dv}\right)=\frac{dR}{de}+a\cos(v-\varpi)\left(\frac{dR}{dv}\right):$$

wherefore, by substituting and dividing all the terms by $\cos(v-\varpi)$,

$$d\varepsilon-d\varpi=-\frac{a(1-e^2)}{e}\cdot\frac{dR}{de}d\zeta-2ra\left(\frac{dR}{dr}\right)d\zeta:$$

and by substituting the value of $d\varpi$, and observing that $\left(\frac{dR}{dr}\right)=\frac{dR}{da}\cdot\frac{a}{r}$, we obtain

$$d\varepsilon=a\sqrt{1-e^2}\left(\frac{1-\sqrt{1-e^2}}{e}\right)\frac{dR}{de}d\zeta-2a^2\frac{dR}{da}\cdot d\zeta. \quad . \quad . \quad . \quad . \quad (17)$$

If the formulas (C) be multiplied, each by its own differential, and the respective results be added, it will be found that the coefficients of $\left(\frac{dR}{dr}\right)$ and $\left(\frac{dR}{dv}\right)$ are each equal to zero, on account of the equations (8) and (10): so that we have,

$$\frac{dR}{da}da+\frac{dR}{de}de+\frac{dR}{d\varepsilon}d\varepsilon+\frac{dR}{d\varpi}d\varpi=0:$$

and this equation will serve to verify the values of da, de, $d\varepsilon$, $d\varpi$, which have been separately investigated.

It remains to examine whether the values of di and dN already found (equation (6)), can be expressed similarly to the other elements. The three quantities N, P, i, or rather the two N and i, since P varies with N, are independent of r and v, and consequently of ζ, a, e, ε, ϖ: wherefore, by differentiating the expressions of x, y, z relatively to i, we shall get

$$\frac{dx}{di}=r\sin(v-P)\sin N\sin i=z\sin N,$$

$$\frac{dy}{di} = -r \sin(v - \text{P}) \cos \text{N} \sin i = -z \cos \text{N},$$

$$\frac{dz}{di} = r \sin(v - \text{P}) \cos i = \frac{r \sin(\lambda - \text{N})}{\sqrt{1 + s^2}}.$$

Let these expressions be multiplied respectively by $\frac{d\text{R}}{dx}$, $\frac{d\text{R}}{dy}$, $\frac{d\text{R}}{dz}$, and then added; the result will be

$$\frac{d\text{R}}{di} = z \left\{ \frac{d\text{R}}{dx} \sin \text{N} - \frac{d\text{R}}{dy} \cos \text{N} \right\} + \frac{d\text{R}}{dz} \cdot \frac{r \sin(\lambda - \text{N})}{\sqrt{1 + s^2}}:$$

and by substituting the values contained in the formulas (B),

$$\frac{d\text{R}}{di} = (1 + s^2) \frac{d\text{R}}{ds} \sin(\lambda - \text{N}) - \frac{d\text{R}}{d\lambda} s \cos(\lambda - \text{N}):$$

and, because $s \cos(\lambda - \text{N}) = \frac{ds}{d\lambda} \sin(\lambda - \text{N})$,

$$\frac{d\text{R}}{di} = \sin(\lambda - \text{N}) \cdot \left\{ (1 + s^2) \frac{d\text{R}}{ds} - \frac{d\text{R}}{d\lambda} \cdot \frac{ds}{d\lambda} \right\}.$$

If the equations (11) be multiplied respectively by $\frac{d\text{R}}{dx}$, $\frac{d\text{R}}{dy}$, $\frac{d\text{R}}{dz}$, and then added, this result will be obtained,

$$\frac{d\text{R}}{di} di + \frac{d\text{R}}{d\text{N}} d\text{N} = 0.$$

By combining this equation and the value of $\frac{d\text{R}}{di}$ with the formulas (6), we get,

$$\left.\begin{aligned} d\text{N} &= \frac{\text{a}}{\sqrt{1 - e^2}} \cdot \frac{1}{\sin i} \cdot \frac{d\text{R}}{di} \cdot d\zeta, \\ di &= -\frac{\text{a}}{\sqrt{1 - e^2}} \cdot \frac{1}{\sin i} \cdot \frac{d\text{R}}{d\text{N}} \cdot d\zeta. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (18)$$

The differentials of the several elements of the orbit of the disturbed planet have now been made to depend upon the function R and its differentials relatively to the elements themselves and to the mean motion ζ. Upon the calculations which this transformation requires, which have long ago been carried as far as human perseverance can well be supposed to go, we do not here enter. The variations of the elements of a disturbed planet, in the most perfect form in which they have been exhibited in the latter part of this paper, are the result of the repeated labours of LAGRANGE and LAPLACE, who, at different times and by different methods, at last succeeded in overcoming the difficulties of this great problem.

In this paper the utmost rigour of investigation has been strictly preserved. No admission or supposition has any where been made for the sake of simplifying calculation or of obtaining a result more readily. The procedure that has been followed likewise makes it easy to change the form of the differentials of the elements of the orbit, as occasion may require. Thus it is obvious from the formulas (C), and from other formulas, that the variations of all the elements may be expressed by means of the three functions $\frac{d\mathrm{R}}{dr}$, $\frac{d\mathrm{R}}{d\lambda}$, $\frac{d\mathrm{R}}{ds}$; or, by means of the three $\frac{d\mathrm{R}}{dx}$, $\frac{d\mathrm{R}}{dy}$, $\frac{d\mathrm{R}}{dz}$; or, by any two of the differentials of R relatively to a, e, ε, ϖ, and one of the two, relatively to i and N; which remark is useful in the theory of the comets.

There is this advantage in expressing the differentials of the elements by means of the function R, that inspection alone discovers the nature of the terms that enter into every formula. But it is not enough to know the form of the terms, we must likewise attend to their convergency. In the present state of the heavens there is no difficulty in this respect, because the eccentricities of the planetary orbits, and their inclinations to the ecliptic are found to be small, and it is upon the smallness of these quantities that the convergency of the series into which R is developed, mainly depends. In the present circumstances of the planetary system, the formulas afford the utmost possible facility for computing the inequalities of the elliptic elements. After all, the inquiry is difficult enough when it is carried beyond a first approximation; for in the second stage of the process every element that enters into a formula being itself a collection of sines or cosines, it is not easy to be assured of the nature of the quantities arising from the combination of so many complex expressions.

If we extend our views and consider the stability of the system of the world, it is necessary to begin with establishing the convergency of the terms into which R is expanded. The mathematical form of these terms will always be the same; but unless their total amount can be estimated with sufficient exactness by a limited number of them, the human understanding can come to no solid decision. Now this will depend upon the effect of the perturbations in changing the eccentricities and the inclinations of the orbits to the ecliptic.

If it can be proved that these elements, after an indefinite lapse of time, will remain of inconsiderable magnitude as they are at present, the convergency of the series will be established, and the form of the terms of which R consists, will enable us to compute the changes in all the elliptic elements, and to decide the great question of the stability of the system. But we cannot enter upon any extended discussion of these points, and shall conclude this paper with some remarks in illustration of the problem we have solved, and of the manner in which we have solved it.

6. If we suppose that there is no disturbing force, or that $R = 0$, we shall have by the equation (1),

$$\frac{1}{a} = \frac{2}{r} - \frac{dx^2 + dy^2 + dz^2}{dt^2 \, . \, \mu};$$

and if V represent the velocity of the planet at the extremity of r, then,

$$V^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2 \, . \, \mu}, \text{ and } \frac{1}{a} = \frac{2}{r} - V^2.$$

This last equation shows that the mean distance a of an elliptic orbit depends only upon the radius vector drawn to any point, and upon the velocity at that point. Conceive that the straight line r extends from the sun in a given direction and to a given length, and from its extremity suppose that a planet is launched into space with the velocity V, the foregoing equation will determine the mean distance a of the immoveable ellipse in which the planet will revolve. The point from which the planet is projected, and consequently r, remaining the same, $\frac{1}{a}$ and V^2 will vary together; and if we suppose that a becomes equal to a, at the same time that V^2 is changed into $V^2 + d \, . \, V^2$ by forces which act continually but insensibly, we shall have these equations,

$$d \, . \, \frac{1}{\mathrm{a}} = - d \, . \, V^2, \quad \text{and } \frac{1}{\mathrm{a}} = \frac{1}{a} - \int d \, . \, V^2.$$

It has been shown that the disturbing forces acting in the directions of x, y, z, and tending to increase these lines, are respectively $\frac{dR}{dx}, \frac{dR}{dy}, \frac{dR}{dz}$: and, by the principles of dynamics, double the sum of the products of these forces, each being multiplied by the element of its direction, is equal to the change effected on the square of the velocity: wherefore,

$$2 \left(\frac{dR}{dx} dx + \frac{dR}{dy} dy + \frac{dR}{dz} dz \right) = 2 \, d'R = d \, . \, V^2;$$

and consequently,

$$d \cdot \frac{1}{\mathrm{a}} = -2\, d' \mathrm{R}, \qquad \text{and} \ \frac{1}{\mathrm{a}} = \frac{1}{a} - 2 \int d' \mathrm{R}.$$

These results agree with the investigation in the fourth section of this paper; and they coincide with the remarkable equation first discovered by LAGRANGE, from which he inferred the invariability of the mean distances and the periodic times of the planets, when the approximation is extended to the first power only of the disturbing force.

It has already been observed that $\frac{d\mathrm{R}}{d\lambda} \cdot \frac{\sqrt{1+s^2}}{r}$ is the disturbing force perpendicular to the plane passing through the sun and the coordinate z, that is, to the planet's circle of latitude; and likewise that $\frac{d\mathrm{R}}{ds} \cdot \frac{1+s^2}{r}$ is the disturbing force in the same plane perpendicular to r the radius vector. The elements of the direction of these forces are respectively $\frac{r\,d\lambda}{\sqrt{1+s^2}}$ and $\frac{r\,ds}{1+s^2}$: wherefore,

$$2\left(\frac{d\mathrm{R}}{d\lambda} d\lambda + \frac{d\mathrm{R}}{ds} ds\right)$$

is the variation produced in the square of the velocity in the direction perpendicular to r. But dv being the small angle described round the sun in the time dt, the space described by the planet perpendicular to r, is $r\,dv$; and consequently $\frac{r\,dv}{dt\sqrt{\mu}}$ is the planet's velocity in that direction. Wherefore, using the symbol δ to denote a variation caused by the disturbing forces perpendicular to the radius vector, and observing that these forces produce no momentary increase or decrease of that line, we get,

$$2\left(\frac{d\mathrm{R}}{d\lambda} d\lambda + \frac{d\mathrm{R}}{ds} ds\right) = \delta \cdot \frac{r^2\,dv^2}{dt^2 \cdot \mu} = r^2\,\delta \cdot \frac{dv^2}{dt^2 \cdot \mu}:$$

consequently,

$$2\,r^2\left(\frac{d\mathrm{R}}{d\lambda} d\lambda + \frac{d\mathrm{R}}{ds} ds\right) = r^4\,\delta \cdot \frac{dv^2}{dt^2 \cdot \mu} = \delta \cdot \frac{r^4\,dv^2}{dt^2 \cdot \mu}:$$

and, as $\frac{r^2\,dv}{dt\sqrt{\mu}}$ and its square vary by no other cause but the action of the forces perpendicular to r, we have

$$2\,r^2\left(\frac{d\mathrm{R}}{d\lambda} d\lambda + \frac{d\mathrm{R}}{ds} ds\right) = d \cdot \frac{r^4\,\mathrm{d}v^2}{\mathrm{d}t^2 \cdot \mu}.$$

Now this is the same differential equation that has already been obtained by a different method in equation (3) of the second section, and from which the value of h^2, the semi-parameter of the variable elliptic orbit, was deduced. That element is therefore as much an immediate deduction from the disturbing forces, as is the mean distance in the equation of LAGRANGE. As the variation of a is the effect of the disturbing force in altering the velocity in the orbit, so the variation of h^2 is the effect of that part of the disturbing force which alters the exact proportionality to the times of the areas described round the sun. The two elements are together sufficient for determining both the form and the magnitude of the momentary elliptic orbit. The placing of this ellipse so as to be in intimate contact with the real path of the planet, a procedure which corresponds to finding the relation between the arcs θ and v, determines the motion of the line of the apsides.

If, lastly, we attend to that part of the disturbing force which is perpendicular to the circle of latitude passing through the planet, and proceed as before, we shall obtain the differential of the equation (4) in the second section. This differential is therefore the effect of the disturbing force in altering the momentary area which is described in the immoveable plane of xy, and which, without the action of this force would be proportional to the time. The elementary area in the immoveable plane is the projection of the area described in the same time in the plane of the orbit; the proportion of the two determines the cosine of the inclination of the variable plane in which the planet moves; and from this it is easy to determine the position of the line of the nodes, as has been fully explained.

What has been said is independent of the nature of the forces in action; and it is obvious that the same method may be applied to estimate the effect of any extraneous force in disturbing the elliptic motion of a planet.

It would appear that in the view we have taken of this problem, we have been making an approach to some general hints contained in the corollaries of the seventeenth proposition of the first book of the Principia. A connexion between the most recondite results of modern analytical science, and the original ideas thrown out by an author who, although he accomplished so much, has unavoidably left much to be supplied by his successors, is undoubtedly worthy of being remarked, and may suggest useful reflections.

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